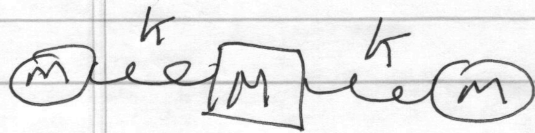
Oscillations and Continua

⇒ Goal is to build up Hamiltonian Theory of Continua / Classical Fields

Prep for — QFT
— Advanced Classical Physics

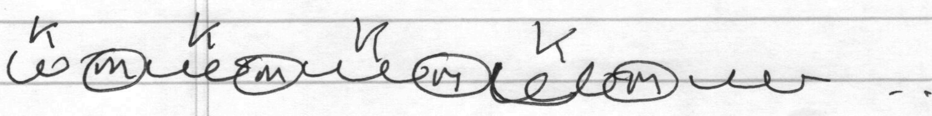
Topics:

→ Small Oscillations / Coupled Oscillators



, etc. (finite)

→ Chains, Continuum Limit



⇒

→ Continuum Dynamics, Waves

includes adiabatic invariants for continua.

and
→ Intro to Elasticity – non-trivial application

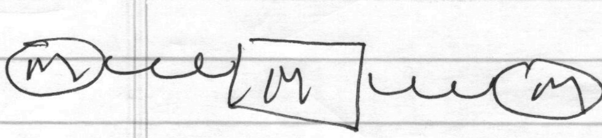
Point:

- normal modes, eigenfrequencies
- energy-momentum theorems
- symmetry (including phase)

$$\begin{cases} LL \to 23, 24 \\ FW \to 4 \end{cases}$$

i.) Small Oscillations

- Consider coupled d-o-f's / i.e. g.c. $q_1, q_2, \ldots q_N$



s/t

$$U = U(q_1, q_2, \ldots q_N)$$

What are collective resonances, modes of system?

then expand near equilibrium:

$$U = U_0 + \sum_j (q_j - q_{j,0}) \left.\frac{\partial U}{\partial q_j}\right|_{q_{j,0}} + \frac{1}{2}\sum_{j,k}(q_j - q_{j,0})(q_k - q_{k,0}) \left.\frac{\partial^2 U}{\partial q_j \partial q_k}\right|_{q_{j,0},\, q_{k,0}} + \ldots$$

So for the minimum:

$$\partial U/\partial q_j = 0 \quad , \quad \det\left|\frac{\partial^2 U}{\partial q_j \partial q_i}\right| > 0$$

then:

$$U = U_0 + \tfrac{1}{2}\sum_{j,k} (q_j - q_{j,0})(q_k - q_{k,0}) \frac{\partial^2 U}{\partial q_j \partial q_k}$$

U_0: irrelevant; $(q_j - q_{j,0})$: x_j; $(q_k - q_{k,0})$: x_k; $\frac{\partial^2 U}{\partial q_j \partial q_k}$: $k_{j,k}$

$$= U_0 + \tfrac{1}{2}\sum_{j,k} x_j\, x_k\, k_{j,k}$$

$k_{j,k}$: stiffness matrix (generally not diagonal)

similarly,

$$T = \frac{1}{2}\sum_{j,k} m_{j,k}\, \dot{x}_j\, \dot{x}_k$$

T: kinetic energy

$m_{j,k}$: mass matrix - generally not diagonal ($\det m_{j,k} > 0$)

so

$$L = \tfrac{1}{2}\sum_{j,k} m_{j,k}\, \dot{x}_j\, \dot{x}_k - \frac{1}{2}\sum_{j,k} k_{j,k}\, x_j\, x_k$$

$$\Rightarrow \frac{d}{dt}\left(\partial L / \partial \dot{x}_j\right) - \frac{\partial L}{\partial x_j} = 0$$

$$\frac{d}{dt}\left(\sum_k m_{j,k}\, \dot{x}_k\right) + \sum_k k_{j,k}\, x_k = 0$$

$$\therefore \boxed{\sum_k \left(m_{j,k}\, \ddot{x}_k + k_{j,k}\, x_k\right) = 0}$$

'Matrix' LEOMs - system

so, extracting overall mass factor:

$$\sum_k \left\{ \lambda_{j,k}\, \ddot{x}_k + \left(\frac{k}{m}\right)_{j,k} x_k \right\} = 0$$

↳ system of linear ODE's $= 0$

note: $k_{j,k} = k_{k,j}$ (order doesn't)

$\lambda_{j,k} = \lambda_{k,j}$

Now, $x_k = A_k\, e^{-i\omega t}$ (vector)

so

normalized mass matrix

$$\sum_k \left(-\omega^2 \lambda_{jk} + \omega^2_{j,k} \right) A_k = 0$$

system of linear eqns.

frequency matrix

→ Homogeneous!

$$\det\left[-\omega^2 \lambda_{j,k} + \omega^2_{j,k} \right] = 0$$

$\Rightarrow \det \left| \omega^2_{j,k} - \chi_{j,k}\,\omega^2 \right| = 0$

$\Rightarrow$ specifies eigenfrequencies / – collective mode frequencies

$\Rightarrow$ ratio of amplitudes $\rightarrow$ eigenvectors

$\Rightarrow$ the solution:

– n eigenfrequencies (ω^2_α, $\alpha = 1, \ldots n$)
– n eigenvectors a^α_j

n.b. degeneracy possible!

so, can write for x_j

$$x_j = \sum_\alpha a^\alpha_j \, e^{-i\omega_\alpha t}$$

$j \rightarrow$ element (i.e. g.c. j)
$\alpha \rightarrow$ eigenvector label

$$\underset{\text{amplitude}}{A^\alpha_j} = \underset{\text{coeff}}{c_\alpha}\,\underset{\text{eigenvector}}{a^\alpha_j}$$

Now, eigenvectors form orthonormal basis

Consider 2 eigenmodes ω_s^2, ω_r^2:
(mode label $\leftrightarrow$ subscript)

$$\omega_s^2 \sum_k \lambda_{j,k} a_k^s = \sum_k \omega_{j,k}^2 a_k^s \qquad (1)$$

and

$$\omega_r^2 \sum_j \lambda_{j,k} a_j^r = \sum_j \omega_{k,j}^2 a_j^r \qquad (2)$$

so

$$\sum_j a_j^r \times (1) - \sum_k a_k^s \times (2) =$$

$$= \sum_{j,k} \Big\{ \omega_s^2 \lambda_{j,k} a_j^r a_k^s - \omega_r \lambda_{j,k} a_j^r a_k^s$$

$$- \sum_{j,k} \lambda_{j,k} \omega_{j,k}^2 \underbrace{\left(a_k^s a_j^r - a_j^r a_k^s \right)}_{0}$$

$$\left(\omega_{j,k}^2 = \omega_{k,j}^2 \right)$$

so

$\Rightarrow$

$$(\omega_s^2 - \omega_r^2) \sum_{k,j} q_j^r q_k^s \lambda_{j,k} = 0$$

or

$$\omega_s^2 \neq \omega_r^2 \Rightarrow \sum_{j,k} \lambda_{j,k} q_j^r q_k^s = 0$$

orthonormality of eigenvectors

normalization $\Rightarrow$

$$\sum_j \lambda_{j,j} q_j^2 = 1$$

so have general orthonormality condition

$$\boxed{\sum_{j,k} \lambda_{j,k} q_k^s q_j^r = \delta_{j,k}} \qquad (*)$$

Can express general oscillation in terms eigenvectors and time dependent amplitudes, i.e.

$$x_j = \sum_\alpha a_j^\alpha \, \eta_\alpha(t)$$

$$c_\alpha e^{-i\omega_\alpha t} \Rightarrow \text{time evolution, amplitude \& phase}$$

orthogonality $\Rightarrow$

$$L = \sum_\alpha \left(\dot{\eta}_\alpha^2 - \omega_\alpha^2 \eta_\alpha^2\right)/2$$ L decomposes

$$\ddot{\eta}_\alpha + \omega_\alpha^2 \eta_\alpha = 0 \qquad \alpha = 1, \dots n$$

η's $\Rightarrow$ normal coords

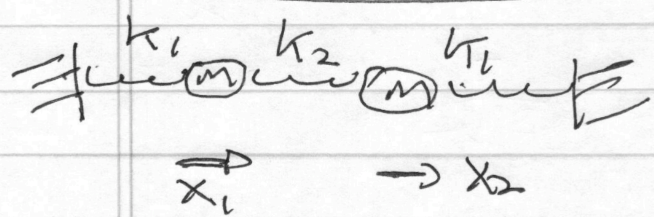
Note: if $\det |\omega_{ij}^2 - \lambda_{ij}\omega^2| = 0$

has double root i.e. $\omega_\alpha^2 = \omega_\beta^2$

$\Rightarrow$ degeneracy! $\Rightarrow$ must arbitrary introduce some condition to det. n orthog eigenvectors

$\Rightarrow$ choice not unique

Some Examples

(i) 

Cranking: G.C.: x_1, x_2

$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) - \left[\frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_1 x_2^2 \right]$$

so

$$m \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$m \ddot{x}_2 = -k_1 x_2 - k_2 (x_2 - x_1)$$

so

$$\ddot{x}_1 + \frac{k_1}{m} x_1 + \frac{k_2}{m} (x_1 - x_2) = 0$$

$$\ddot{x}_2 + \frac{k_1}{m} x_2 - \frac{k_2}{m} (x_1 - x_2) = 0$$

or

$$\omega_0^2 = (k_1 + k_2)/m$$

$$\ddot{x}_1 + \omega_0^2 x_1 - \frac{k_2}{m} x_2 = 0$$

$$\ddot{x}_2 + \omega_0^2 x_2 - k_2/m \; x_1 = 0$$

$$x_1 = A e^{-i\omega t}$$

$$x_2 = B e^{-i\omega t}$$

$$(\omega_0^2 - \omega^2) A - (k_2/m) B = 0$$

$$-(k_2/m) A + (\omega_0^2 - \omega^2) B = 0$$

$$(\omega^2 - \omega_0^2)^2 - (k_2/m)^2 = 0$$

$\Rightarrow$

$$\omega^2 = \omega_0^2 \pm k_2/m$$

$$\omega^2 = k_1/m$$

$$\omega^2 = k_1/m + 2k_2/m$$

(eigenfrequencies)

$A, B \Rightarrow$ eigenvectors

$$\omega^2 = \omega_0^2 - k_2/m = k_1/m$$

$$+k_2/m\ A \quad - k_2/m\ B = 0$$

$$-k_2/m\ A \quad + k_2/m\ B = 0$$

so

$$A = B, \quad a_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} / \sqrt{2}$$

normalization

$$\omega^2 = \omega_0^2 + k_2/m$$

$$-k_2/m\ A \quad -(k_2/m)\ B = 0$$

$$-k_2/m\ A \quad - k_2/m\ B = 0$$

$$A = -B, \quad a_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} / \sqrt{2}$$

so

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{c_1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_+ t}$$

high freq $\Rightarrow$ anti-symmetric

$$+ \frac{c_2}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_- t}$$

low freq $\Rightarrow$ symmetric

Of course, could: diffrnt notation

$$\ddot{x}_1 + \omega_0^2 x_1 - (k_2/m)\, x_2 = 0$$

$$\ddot{x}_2 + \omega_0^2 x_2 - (k_2/m)\, x_1 = 0$$

– add $\quad x_+ = x_1 + x_2$

$$\ddot{x}_+ + \omega_0^2 x_+ - k_2/m \;\; x_+ = 0$$

$x_+ = x_1 + x_2$ $\qquad \ddot{x}_+ + (k_1/m)\, x_+ = 0$

$\omega_+^2 = k/m$ $\qquad \eta_+ = x_+$

– subtract

$$\ddot{x}_- + \omega_0^2 x_- + (k_2/m)\, x_- = 0$$

$$\eta_- = x_-$$

$x_- = x_1 - x_2$

$\omega_-^2 = (k_1 + 2k_2)/m$

$\to$ splits frequencies

ii.) $V = -\alpha x y$ interaction of 2 oscillators

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}k(x^2+y^2) + \alpha x y$$

$$m\ddot{x} + kx - \alpha y = 0$$

$$m\ddot{y} + kx - \alpha x = 0$$

so

$$\ddot{x} + \omega_0^2 x - \alpha/m\, y = 0$$

$$\ddot{y} + \omega_0^2 y - \alpha/m\, x = 0$$

so

$$\eta_\pm = x \pm y$$

$$\ddot{\eta}_+ + \omega_0^2 \eta_+ - \alpha/m\, \eta_+ = 0$$

$$\ddot{\eta}_- + \omega_0^2 \eta_- + \alpha/m\, \eta_- = 0$$

$$\omega_+^2 = \omega_0^2 - \alpha/m \qquad \omega^2 = (k-\alpha)/m$$

symm $\to$ low freq.

$$\omega_- = \omega_0^2 + \alpha/m = (k+\alpha)/m$$

anti-symm $\to$ high ω.

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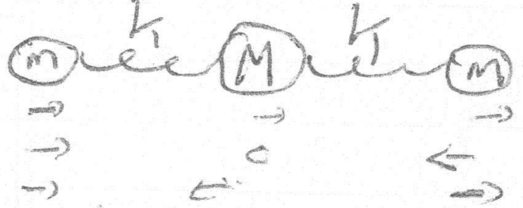
symmetry $\Leftrightarrow$ zero frequency mode!
(why $\Leftrightarrow$ displace with <u>no</u> change in energy)

iii (c)



$\omega^2 = 0$
$\omega^2 = k/m$ (more symm)
$\omega^2 = k\mu/mM$. (less symm)

<u>Key point</u> $\Rightarrow$ no external forces, so CM constant $\Rightarrow$ 1 degree symmetry.

$\Rightarrow$

(*) $$m x_1 + M x_2 + m x_3 = \text{const} = 0$$

$\Rightarrow$ reduce $3\times3 \Rightarrow 2\times2$

$$L = \tfrac{1}{2}\left(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2\right) - \tfrac{1}{2}k(x_2 - x_1)^2 - \tfrac{1}{2}k(x_3 - x_2)^2$$

but $x_2 = -\dfrac{m}{M}(x_1 + x_3)$

$\Rightarrow$

$$L = \tfrac{1}{2}\left[m(\dot{x}_1^2 + \dot{x}_3^2) + M\frac{m^2}{M^2}(\dot{x}_1 + \dot{x}_3)^2\right]$$
$$- \tfrac{1}{2}k\left(-\frac{m}{M}(x_1 + x_3) - x_1\right)^2 - \tfrac{1}{2}k\left(x_3 + \frac{m}{M}(x_1 + x_3)\right)^2$$

etc.

(*)
Guessing the modes $\leftrightarrow$ symmetry:

$1/\mu = \frac{1}{m} + \frac{1}{m} + \frac{1}{M}$

$\omega^2 = 0$; translation mode

$\omega^2 = k/m$; symmetric mode $\rightarrow \eta = x_1 - x_3$

$\omega^2 = k\mu/mM$; anti-symmetric mode $\rightarrow \eta = x_1 + x_3$

→ Aside: Example from Continuum
Translation and Zero Frequency Modes

Consider:

$$L = \int_{x_1}^{x_2} dx\, \mathcal{L}$$

$$\mathcal{L} = \frac{(\partial_t f)^2}{2} - \frac{(\partial_x f)^2}{2} - U(f)$$

i.e. $U = 0$ → wave equation
$U = f^2/2$ → Klein-Gordon
etc.
$U = \alpha f^2/2 + \frac{\beta f^4}{4}$ → ϕ^4 model 1D.
(can relate magnetism).

∞ for NL string:

$$\text{LEM} \Rightarrow \quad \partial_t^2 f - \partial_x^2 f + \frac{\partial U}{\partial f} = 0$$

For static, eqblm solution:

$$\partial_t^2 f_0 = 0$$

$$\Rightarrow \quad \partial_x^2 f_0 - \frac{\partial U}{\partial f_0} = 0$$

Now, for fluctns. about:

$$f = f_0 + \tilde{f}$$

$$\tilde{f} = \hat{f} e^{-i\omega t} \qquad \hat{f} = \hat{f}(x)$$

$\Rightarrow$ plug onto EOM and linearize:

$$-\omega^2 \hat{f} - \partial_x^2 \hat{f} + \frac{\partial U}{\partial f}(f_0 + \hat{f}) = 0$$

$$\Rightarrow$$

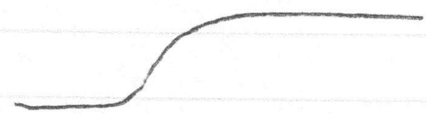
$$-\omega^2 \hat{f} - \partial_x^2 \hat{f} + \frac{\partial^2 U}{\partial f^2} \hat{f} = 0$$

i.e. $$-\partial_x^2 \hat{f} + \left(\frac{\partial^2 U}{\partial f_0^2}\right) \hat{f} = \omega^2 \hat{f}$$ eigenmode eqn.

note $\omega^2 = 0 \Rightarrow -\partial_x^2 \hat{f} + \left(\frac{\partial^2 U}{\partial f_0^2}\right) \hat{f} = 0$

but can also observe:

$$-\partial_x^2 f_0 + \frac{\partial U}{\partial f_0} = 0$$



$\Rightarrow$ a solution $f_0(x)$.

Now can translate that solution arbitrarily, as have translation symmetry in x

i.e. $f_0(x) \to f_0(x + \delta x_0)$ must be solution

δx_0: infinitesimal centroid shift

$$-\partial_x^2 \left(f_0(x+\delta x_0)\right) + \frac{\partial U}{\partial f}\left(f_0(x+\delta x_0)\right) = 0$$

expand in δx_0:

$$\delta x_0 \, (-\partial_x^3 f_0) + \delta x_0 \, \frac{\partial^2 U}{\partial f^2}(\partial_x f_0) = 0$$

i.e. $$\frac{d}{dx}\left(-\partial_x^2 f_0 + \frac{\partial U}{\partial f_0}\right) = 0 .$$

so

$$-\partial_x^2(\partial_x f_0) + \left.\frac{\partial^2 V}{\partial f^2}\right|_{f_0} (\partial_x f_0) = 0$$

$\Leftrightarrow$ but eigenmode eqn is:

$$-\partial_x^2 \hat{f} + \left.\frac{\partial^2 V}{\partial f^2}\right|_{f_0} \hat{f} = \omega^2 \hat{f}$$

so

$\omega^2 = 0$ is eigenmode with eigenfunction $\partial_x f_0$

→ translation mode, due translation symmetry of $\mathcal{L}$.

→ obviously generalizable.

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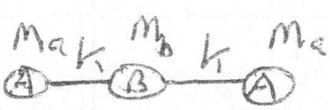
iii.) Triatomic Molecule → 2D

$\underline{3 \times 2} = 6$ modes

a) Linear



{ harmonic binding
{ 1D → as previous example.

modes : 1) $\omega^2 = 0$ → translation x, y

$\underline{3}$ symm. → rotation with m_b fixed

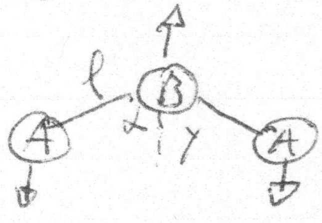
{ i.e. 3 invariant transformations
{ ⇒ $\underline{\underline{3}}$ zero frequency modes

(vibration) 2) linear; symmetric $\omega^2 = k/m$; $(\cancel{x_1 - x_3})$

$\underline{2}$ antisymmetric $(x_1 + \cancel{x_3})$; $kM/m_A m_B$

(rotation) 3) bending → symmetric in x

$\underline{1}$



(antisymmetric → rotation)

⇒ anti-symmetry y

(Ycm inv.).

Proceeding as before:

$$m_A y_1 + m_B y_2 + m_A y_3 = 0$$

and $y_1 = y_2$ (symmetry)

$$T = \frac{1}{2} m_A (\dot{y}_1^2 + \dot{y}_3^2) + \frac{m_B}{2} \dot{y}_2^2$$

↳ can eliminate in terms y_1, y_3

bend of molecule

$U = \frac{1}{2}k(\delta L)^2$; $\delta L = l_1 \cos\alpha_1 + l_2 \cos\alpha_2$

$l_1 = l_2$

small osc $= l\left[\frac{(y_1 - y_2)}{l} + \frac{(y_3 - y_2)}{l}\right]$

$\Rightarrow$

$$L = \frac{1}{2} m_A (\dot{y}_1^2 + \dot{y}_3^2) + \frac{1}{2} m_B \dot{y}_2^2 - \frac{1}{2} k \left[(y_1 - y_2) + (y_3 - y_2)\right]^2$$

subst for y_2

$$= \frac{m_A m_B}{4M} l^2 \dot{\delta}^2 - \frac{1}{2} k l^2 \delta^2$$

$$\delta = (y_1 + y_3 - 2y_2)/l$$

so $\boxed{\omega^2 = 2kM / m_A m_B.}$